Continuous selections and prime numbers

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Abstract

Let (X,τ) be a Hausdorff space and $n\in\omega$. We prove that if X admits a continuous selection over $\mathcal{F}_n(X)$ (the nonempty subsets of X of cardinality at most n), then for every $n\leq m\leq 2n$ such that m is not a prime number, X admits a continuous selection over $[X]^m$ (the subsets of X of cardinality m). As a consequence of this, a space X admits a continuous selection for every natural number if and only if the same is true for every prime number.

1. Preliminaries

Our notation is fairly standard. $[X]^n$ and $[X]^{<\omega}$ will denote the set of all subsets of X of cardinality n and the set of all finite subsets of X respectively. If (X,τ) is a topological space, $\mathcal{F}(X)$ will be the set of all nonempty closed subsets of X, and if $n\in\omega$, then $\mathcal{F}_n(X):=\{A\in\mathcal{F}(X)\mid |A|\leq n\}$. We can endow $\mathcal{F}(X)$ with the topology generated by the sets

$$\langle\mathcal{V}\rangle:=\{A\in\mathcal{F}(X)\mid \forall V\in\mathcal{V}(A\cap V\neq\emptyset)\wedge A\subseteq\bigcup\mathcal{V}\}$$

where $\mathcal{V}$ is a finite subset of τ. We will refer to this topology as the Vietoris topology τ_V.

All the spaces considered here are at least T_2.

The study of continuous selections began in the early 1950's by E. Michael [8]. Since then, there has been an impressive amount of research that seems far from ending in the near future.

Definition 1. *Given $\mathfrak{F}\subseteq\mathcal{F}(X)$, we say that $f:\mathfrak{F}\longrightarrow X$ is a selection if $f(A)\in A$, for every $A\in\mathfrak{F}$. Moreover, f is a continuous selection if it is a selection which is continuous with respect to the Vietoris topology restricted to $\mathfrak{F}$. Additionally, given $n\in\omega$ we define*

- $Sel_n(X):=\{f:[X]^n\longrightarrow X\mid f \text{ is a selection}\}$.
- $Sel_{\leq n}(X):=\{f:\mathcal{F}_n(X)\longrightarrow X\mid f \text{ is a selection}\}$.
- $Sel^c_n(X):=\{f:[X]^n\longrightarrow X\mid f \text{ is a continuous selection}\}$.

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- $Sel^c_{\leq n}(X) := \{f : \mathcal{F}_n(X) \longrightarrow X \mid f \textit{ is a continuous selection}\}$.

Selecting and ordering are two closely related concepts. Examples of this can be found in [8] where it is proven that in the connected spaces, $Sel^c_2(X) \neq \emptyset$ is equivalent to weak orderability of X; or in [9], where it is proven that the same is true for compact spaces. In [9] it was asked if spaces for which $Sel^c_2(Y) \neq \emptyset$ are exactly those that are weakly orderable. A negative answer to that question was given in [7] by constructing an almost disjoint family $\mathcal{A}$ for which $Sel^c_2(\Psi(\mathcal{A})) \neq \emptyset$ but $\Psi(\mathcal{A})$ is not weakly orderable. Here, $\Psi(\mathcal{A})$ is the associated Isbell-Mrówka space. From this, we can conclude that although these concepts behave similar in many ways, they are not equivalent. Because of this, it is natural to ask how close are spaces for which $Sel^c_2(X) \neq \emptyset$ to being weakly orderable. For example, if Y is a weakly orderable space, then for every $n > 1$ we will have that $Sel^c_{\leq n}(Y) \neq \emptyset$. So in general, what can we say of $Sel^c_n(X)$, knowing that $Sel^c_2(X) \neq \emptyset$? The following question was posed in [5].

Problem 1 ([5]). *Does there exist a space X such that $Sel^c_2(X) \neq \emptyset$, but $Sel^c_{\leq n}(X) = \emptyset$ for some $n > 2$?*

This question is still open even in the case $n = 3$. In [2], it is proven that if $Sel^c_2(X) \neq \emptyset$, then it is equivalent that $Sel^c_3(X) \neq \emptyset$ and $Sel^c_{\leq 3}(X) \neq \emptyset$. This result was later generalized in [3] by the following Theorem.

Theorem 1 ([3]). *Let $n \in \omega$ be such that $Sel^c_{\leq n}(X) \neq \emptyset$. The following statements are equivalent:*

a) $Sel^c_{\leq n+1}(X) \neq \emptyset$.

b) $Sel^c_{n+1}(X) \neq \emptyset$.

By means of Theorem 1, we can restate Problem 1 as:

Problem 2. *Does there exist a space X for which $Sel^c_2(X) \neq \emptyset$, but $Sel^c_n(X) = \emptyset$ for some $n > 2$?*

A partial answer to this question can be found [7], where the next Theorem shows up.

Theorem 2 ([7]). *Let X be a separable space for which $Sel^c_2(X) \neq \emptyset$. Then, there exists an orderable space L and a continuous map $f : X \longrightarrow L$ such that $|f^{-1}[\{y\}]| \leq 2$ for every $y \in L$. In particular, X admits a continuous selection over $[X]^{<\omega} \backslash \{\emptyset\}$.*

In [6] it is proven that $Sel^c_4(X) \neq \emptyset$ provided that $Sel^c_{\leq 3}(X) \neq \emptyset$. This result was later generalized in [3] by means of the following Theorem.

Theorem 3 ([3]). *Supose that X is a space for which $Sel^c_{\leq 2n+1}(X) \neq \emptyset$ for some $n \geq 1$. Then $Sel^c_{2n+2}(X) \neq \emptyset$.*

So in particular, if Problem 2 has an affirmative answer, there must exist X for which $Sel_2^c(X) \neq \emptyset$ but $Sel_n^c(X) = \emptyset$ for some odd number n. Moreover, if we aim to answer Problem 2, we may only worry about $Sel_n^2(X)$ for odd numbers n.

The aim of this paper is to generalize Theorem 3 by showing that $Sel_{\leq n}^c(X) \neq \emptyset$ implies $Sel_m^c(X) \neq \emptyset$ for every $n \leq m \leq 2n$ which is not a prime number, see Theorem 4. This will be based on the following considerations about special Vietoris neighborhoods preserving selection-relations.

2. Families preserving selection-relations

The core of the following proposition can be found in [3], or in [6] for the special case where $\mathfrak{F} = [X]^2$. For the general case of $\mathfrak{F} = [X]^n$, the reader may consult [1].

Proposition 1. *Let $\mathfrak{F} \subseteq [X]^{<\omega} \backslash \{\emptyset\}$, and $f : \mathfrak{F} \longrightarrow X$ a selection. The following statements are equivalent:*

a) f is continuous.

b) For every $S \in \mathfrak{F}$ there is a finite family of pairwise disjoint open sets $\mathcal{U}$, such that $|S| = |\mathcal{U}|$, $S \in \langle \mathcal{U} \rangle$ and $f[\langle \mathcal{U} \rangle] \subset V$ for some $V \in \mathcal{U}$.

Proof. To prove $a)$ implies $b)$, take $S \in \mathfrak{F}$. Since X is T_2, there is a finite parwise disjoint family of open sets $\mathcal{W}$, such that $S \in \langle \mathcal{W} \rangle$ and $|S| = |\mathcal{W}|$. Take $W \in \mathcal{W}$ such that $f(S) \in W$. Since f is continuous there is a finite family of open sets $\mathcal{V}$, such that $S \in \langle \mathcal{V} \rangle$ and $f[\langle \mathcal{V} \rangle] \subseteq W$. The desired set is $\mathcal{U} = \{\bigcap\{V \in \mathcal{V} \cup \mathcal{W} \mid s \in V\} \mid s \in S\}$.

To prove $b)$ implies $a)$, take $S \in \mathfrak{F}$ and W an open set such that $f(S) \in W$. It should be clear that if we take $\mathcal{U}$ as in the hypotheses, and let $V \in \mathcal{U}$ such that $f(S) \in V$ then $\mathcal{V} = (\mathcal{U} \backslash \{V\}) \cup \{W \cap V\}$ is such that $S \in \langle \mathcal{V} \rangle$ and $f[\langle \mathcal{V} \rangle] \subseteq W$. Hence, f is continuous. □

Definition 2. *Let $n \geq k \geq 2$, $\mathfrak{F} \subseteq \mathcal{F}_n(X)$, $f : \mathfrak{F} \longrightarrow X$ a selection, and $\mathcal{U}$ a finite pairwise-disjoint family of nonempty open subsets of X. We will say that $\mathcal{U}$ preserves (f, k)-relations if $|\mathcal{U}| \geq k$ and for every $\mathcal{V} \in [\mathcal{U}]^k$ there exists some $V \in \mathcal{V}$ such that*

$$f[\langle \mathcal{V} \rangle \cap [X]^k] \subseteq V.$$

In general we will say that $\mathcal{U}$ preserves f-relations if and only if $\mathcal{U}$ preserves (f, k)-relations for every $n \geq k \geq 2$.

Proposition 2. *Let $n \geq 2$, and $f \in Sel_{\leq n}^c(X)$(or $f \in Sel_n^c(X)$). For every $m \geq n$ and for every $S \in [X]^m$ there is a finite family of pairwise disjoint open sets $\mathcal{U}$, such that $S \in \langle \mathcal{U} \rangle$, $|\mathcal{U}| = m$ and $\mathcal{U}$ preserves f-relations(respectively, (f, n)-relations).*

Proof. For every $Z \in \mathcal{F}_n(S)$ apply Proposition 1 to find a pairwise disjoint family of open sets $\mathcal{U}_Z$, such that $Z \in \langle \mathcal{U}_Z \rangle$, $|Z| = |\mathcal{U}_Z|$ and $f[\langle \mathcal{U}_Z \rangle] \subseteq V$ for some $V \in \mathcal{U}_Z$. For every $s \in S$ define

$$U_s = \bigcap \{V \mid \text{ there is } Z \in \mathcal{F}_n(S) \text{ such that } V \in \mathcal{U}_Z \text{ and } s \in Z \cap V\}.$$

It is now evident that $\mathcal{U} = \{U_s \mid s \in S\}$ is as required. $\square$

3. Isomorphisms of Selections

Isomorphisms of selections were introduced in [7] for the special case of selections over $[X]^2$. Now, we present the natural generalization of that notion.

Definition 3. *Given Z and Y arbitrary sets, $n \geq 2$, $f \in Sel_n(Z)$, and $g \in Sel_n(Y)$. We will say that f and g are isomorphic if there is a bijection $\phi : Z \longrightarrow Y$ such that for every $S \in [Z]^n$ it happens that*

$$g(\phi[S]) = \phi(f(S)).$$

In this case, we will say that ϕ is an isomorphism between f and g.

For the remaining of this section, we will fix for every $m \in \omega$, a set Y_m of m elements.

Definition 4. *Let $m, k \geq n \geq 2$, $f \in Sel_{\leq k}(X)$ and $g \in Sel_n(Y_m)$. We define*

$$\mathcal{P}(g) := \{S \in [X]^m \mid f|_{[S]^n} \text{ is isomorphic to } g\}.$$

Notice that the previous sets are just the equivalence classes of the equivalence relation $\sim$ over $[X]^m$ defined as $S \sim Z$ if and only if $f|_{[S]^n}$ is isomorphic to $f|_{[Z]^n}$.

In the next two propositions we will show that each $\mathcal{P}(g)$ is clopen.

Proposition 3. *Let $m, k \geq n \geq 2$, $f \in Sel_{\leq k}(X)$ and $\mathcal{U}$ be a pairwise-disjoint family of m nonempty subsets. If $\mathcal{U}$ preserves (f, n)-relations, then for every $Y, Z \in \langle \mathcal{U} \rangle \cap [X]^m$ the function $\phi : Z \longrightarrow Y$ defined as $\phi(z) \in Y \cap U$ whenever $U \in \mathcal{U}$ and $z \in Z \cap U$ is an isomorphism between $f|_{[Z]^n}$ and $g|_{[Y]^n}$.*

Proof. Since $Y, Z \in \langle \mathcal{U} \rangle \cap [X]^m$ and the elements of $\mathcal{U}$ are pairwise disjoint, is easy to see that ϕ is well defined and bijection between Z and Y. To see that ϕ is an isomorphism take $S \in [Z]^n$. Let $\mathcal{U}_S := \{U \in \mathcal{U} \mid S \cap U \neq \emptyset\}$, and note that $\phi[S] \in \langle \mathcal{U}_S \rangle$. Since $\mathcal{U}_S \in [\mathcal{U}]^n$, and $\mathcal{U}$ preserves (f, n)-relations then there is $V \in \mathcal{U}_S$ such that $f[\langle \mathcal{U}_S \rangle \cap [X]^n] \subseteq V$. In particular $f(S), f(\phi[S]) \in V$, so by definition of ϕ, we have that $\phi(f(S)) = f(\phi[S])$.

$\square$

Proposition 4. *Let $m, k \geq n \geq 2$, $f \in Sel^c_{\leq k}(X)$ and $g \in Sel_n(Y_m)$. Then $\mathcal{P}(g)$ is clopen in $[X]^m$.*

Proof. Since $\{\mathcal{P}(g) \mid g \in Sel_n(Y_m)\}$ form a partition of $[X]^m$, it's enough to proof that for every $g \in Sel_n(Y_m)$, $\mathcal{P}(g)$ is open.

Take $g \in Sel_n(Y_m)$ and $S \in \mathcal{P}(g)$. Using Proposition 2, we find a finite family of pairwise disjoint open sets $\mathcal{U}$, such that $S \in \langle \mathcal{U} \rangle$, $|\mathcal{U}| = m$ and $\mathcal{U}$ preserves (f, n)-relations. To finish just notice that by Proposition 3 we will have that $S \in \langle \mathcal{U} \rangle \subseteq \mathcal{P}(g)$. $\square$

Definition 5. *For an arbitrary Y, let $n \geq 2$, $g \in Sel_n(Y)$ and $x \in Y$. We define*

$$\mathcal{W}_g(x) := |g^{-1}(x)|.$$

Additionally, if $k \in \omega$ we define

$$\mathcal{Q}_g(k) := \{x \in Y \mid \mathcal{W}_g(x) = k\}.$$

If g is clear by context, we will omit it.

It is easy to see that these notions are preserved by isomorphisms, we will use this fact in the following lemma.

Lemma 1. *Let $m, k \geq n \geq 2$, $f \in Sel^c_{\leq k}(X)$ and $g \in Sel_n(Y_m)$. If there are $x, y \in Y_m$ such that $\mathcal{W}_g(x) \neq \mathcal{W}_g(y)$ and $\frac{m}{2} \leq k$ then there exists a continuous selection h over $\mathcal{P}(g)$.*

Proof. The set $\mathcal{Q} = \{\mathcal{Q}_g(r) \mid r \in \omega \text{ and } \mathcal{Q}_g(r) \neq \emptyset\}$ is a partition of Y_m and by hypothesis it has at least two elements. Thus, there exists $r \in \omega$ such that $\mathcal{Q}_g(r) \neq \emptyset$ and $|\mathcal{Q}_g(r)| \leq \frac{m}{2}$. Take $r_0 \in \omega$ the least natural with this property and let $k_0 := |\mathcal{Q}_g(r_0)|$.

We define $h : \mathcal{P}(g) \longrightarrow X$ given by $h(Z) = f(\mathcal{Q}_{f|_{[Z]^n}}(r_0))$. We know h is well defined since for every $Z \in \mathcal{P}(g)$ it happens that $|\mathcal{Q}_{f|_{[Z]^n}}(r_0)| = |\mathcal{Q}_g(r_0)| = k_0 \leq k$. To prove that h is continuous, take $Z \in \mathcal{P}(g)$. Applying Proposition 2, we can find $\mathcal{U}$ a pairwise-disjoint family of m open sets, such that $Z \in \langle \mathcal{U} \rangle$ and $\mathcal{U}$ preserves f-relations. Finally take $V \in \mathcal{U}$ such that $h(Z) \in V$.

For $Y \in \langle \mathcal{U} \rangle \cap \mathcal{P}(g)$, let $\phi : Z \longrightarrow Y$ be function defined in Proposition 3. Since $\mathcal{U}$ preserves f-relations, in particular we will have that:

a) ϕ is an isomorphism from $f|_{[Z]^n}$ to $f|_{[Y]^n}$,

b) ϕ is an isomorphism from $f|_{[Z]^{k_0}}$ to $f|_{[Y]^{k_0}}$.

By *a)* we will have that $\phi[\mathcal{Q}_{f|_{[Z]^n}}(r_0)] = \mathcal{Q}_{f|_{[Y]^n}}(r_0)$. Moreover, if

$$\mathcal{U}_Z = \{U \in \mathcal{U} \mid U \cap \mathcal{Q}_{f|_{[Z]^n}}(r_0) \neq \emptyset\}$$

then $\mathcal{Q}_{f|_{[Z]^n}}(r_0) \in \langle \mathcal{U}_Z \rangle$. By *b)* this implies that $\phi(h(Z)) = \phi(f(\mathcal{Q}_{f|_{[Z]^n}}(r_0))) = f(\phi[\mathcal{Q}_{f|_{[Z]^n}}(r_0)]) = f(\mathcal{Q}_{f|_{[Y]^n}}(r_0)) = h(Y)$, so by definition of ϕ we get that $h(Y) \in V$. Since Y was arbitrary we conclude that $h[\langle \mathcal{U} \rangle] \subseteq V$, so by Proposition 1 we are done.

$\square$

Lemma 2. *let $k, m \geq 2$ and $f \in Sel^c_{\leq k}(X)$. If $\frac{m}{2} \leq k$, and there is $2 \leq n \leq k$ such that for every $Z \in [X]^m$ there exists $x, y \in Z$ with $\mathcal{W}_{f|_{[Z]^n}}(x) \neq \mathcal{W}_{f|_{[Z]^n}}(y)$, then $Sel^c_m(X) \neq \emptyset$.*

Proof. By hypothesis, for every $g \in Sel_n(Y_m)$ such that $\mathcal{P}(g) \neq \emptyset$ the hypotheses of Lemma 1 are fulfilled. Thus there exists a continuous selection $h_g : \mathcal{P}(g) \longrightarrow X$. Since $\mathcal{P}(g)$ is clopen, then

$$h = \bigcup \{h_g \mid g \in Sel_n(Y_m) \text{ and } \mathcal{P}(g) \neq \emptyset\}$$

is a continuous selection over $[X]^m$. □

Lemma 3. *Let $m \geq p$ such that that p is prime. If there exists $g \in Sel_p(Y_m)$ such that $\mathcal{W}(x) = \mathcal{W}(y)$ for every $x, y \in Y_m$, then p does not divide m.*

Proof. Le g be as in the hypothesis. Suppose toward a contradiction that p divides m, and let k be such that for every $x \in Y_m$ it happens that $k = \mathcal{W}(x)$. Notice that

$$mk = \sum_{x \in Y} \mathcal{W}(x) = |[Y_m]^p| = \frac{m!}{p!(m-p)!}.$$

Thus $pk = \frac{(m-1)!}{(p-1)!(m-p)!}$, so in particular p divides $\frac{(m-1)!}{(m-p)!}$. Since p is prime there must exist $1 \leq j < p$ such that p divides $m - j$, but p divides m so this means that p divides j, which is impossible since $j < p$. We conclude that p does not divides m. □

Theorem 4. *Let $p, k, m \geq 2$ with p prime, and suppose $Sel^c_{\leq k}(X) \neq \emptyset$. If $p, \frac{m}{2} \leq k$ and p divides m, then $Sel^c_m(X) \neq \emptyset$.*

Proof. Let $f \in Sel^c_{\leq k}(X)$. Since p divides m, applying Lemma 3 we conclude that for every $Z \in [X]^p$ there are $x, y \in Z$ such that $\mathcal{W}_{f|_{[Z]^n}}(x) \neq \mathcal{W}_{f|_{[Z]^n}}(y)$, thus k, m and f fulfill the hypothesis of Lemma 2, and in consequence $Sel^c_m(X) \neq \emptyset$. □

Note that Theorem 4 is never optimal in the sense that Bertrand Postulate implies that there is always a prime m between k and $2k$, so in this case p would not divide m for any $p \leq k$.

Corollary 1. *Let $2 \leq n$ and suppose that $Sel^c_{\leq n}(X) \neq \emptyset$. If $n + 1$ is not prime, then $Sel^c_{n+1}(X) \neq \emptyset$.*

Theorem 5. *Let X be a T_2 topological space. The following statements are equivalent:*

a) For every $1 < n \in \omega$, $Sel^c_n(X) \neq \emptyset$.

b) For every prime p, $Sel^c_p(X) \neq \emptyset$.

Proof. $a) \rightarrow b)$ trivial.

$b) \rightarrow a)$ By induction. Let $1 < n \in \omega$ and suppose that for every $1 < m < n$ we know that $Sel^c_n(X) \neq \emptyset$. If n is prime, we use the hypothesis . If n is not prime, then we make use of Theorem 1 to conclude that $Sel^c_{\leq n-1}(X) \neq \emptyset$. Thus, by Corollary 1 it follows that $Sel^c_n(X) \neq \emptyset$. □

In [1] it is proved that for every $n \geq 2$, there is a connected second countable space X such that $Sel^c_{n+1}(X) \neq \emptyset$ but $Sel^c_k(X) = \emptyset$ for every $k \leq n$, so in general, the existence of continuous selections for big subsets of X does not imply the existence of continuous selections for small subsets of X. Nevertheless, we can ask the following.

Problem 3. *Suppose that $m > n \geq 2$. Does the existence of a continuous selection $f : \mathcal{F}_m(X)\backslash\mathcal{F}_n(X) \longrightarrow X$ imply that $Sel^c_k(X) \neq \emptyset$ for some $k > m$?*

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